# Hopf superalgebra contractions and R-matrix for fermions



1991 J. Phys. A: Math. Gen. 24 5675

(http://iopscience.iop.org/0305-4470/24/24/006)

J. Phys. A: Math. Gen. 24 (1991) 5675–5682. Printed in the UK

# Hopf superalgebra contractions and $R$-matrix for fermions

E Celeghini†‖, R Giachetti‡¶, P P Kulish§, E Sorace† and M Tarlini†

† Dipartimento di Fisica, Università di Firenze and INFN–Firenze, Italy

‡ Dipartimento di Matematica, Università di Bologna, Italy

§ Leningrad branch of Steklov Mathematical Institute, USSR

Received 1 July 1991

**Abstract.** Different contractions are applied to the quantum superalgebra $\mathrm{osp}_q(1|2)$. In the first of them the graded analogue s-$\mathrm{H}_q(1)$ of the one-dimensional Heisenberg quantum algebra is obtained and its $R$-matrix explicitly calculated. In the second contraction a $\mathrm{Z}_2$-graded version of the $q$-oscillator is proposed and finally the supersymmetric two-dimensional Euclidean quantum algebra s-$\mathrm{E}_q(2)$ is found.

## 0. Introduction

In recent times increasing interest has been shown in the study of quantum algebras and to their physical applications. The most commonly considered cases are related to the quantized version of simple Lie algebras, where a general theory has been developed which rephrases the classical Cartan theory and allows the calculation of fundamental objects such as the $R$-matrix [1]. The classical concepts and procedures have been generalized to graded structures where, in particular, actions of groups and algebras have been studied [2]. Taking into account these interests, quantum analogues of simple superalgebras have been defined [3].

A general theory for non-semisimple algebras is not available. A popular method for collecting information on such structures, starting from semisimple ones, is the contraction technique [4] which has also been generalized to the graded case [5]. The contraction for $q$-deformed structures was introduced in [6, 7] and the quantization of the Heisenberg and Euclidean Hopf algebras was attained in [8, 9].

The purpose of this article is to give the quantum superalgebras obtained by different contractions of $\mathrm{osp}_q(1|2)$. In the first of them we obtain the quantum superalgebra s-$\mathrm{H}_q(1)$ with one boson and one fermion generator together with a number operator; the $R$-matrix of this Hopf algebra is explicitly calculated. In the second contraction the quantum parameter is fixed and we obtain the $\mathbf{Z}_2$-graded version of the $q$-oscillator [10, 11]. A final contraction leads to the Hopf algebra s-$\mathrm{E}_q(2)$, the supersymmetric analogue of the construction given in [6, 7, 9].

A preliminary version of some of the results obtained in this article was presented during the first semester on 'Quantum Groups' (Leningrad, 1990) [9].

‖ E-mail: celeghini@fi.infn.it

¶ E-mail: giachetti@dm.unibo.it

0305-4470/91/245675+08\$03.50  

## 1. Quantum superalgebra s-$\mathrm{H}_q(1)$

The quantum superalgebra $\mathrm{osp}_q(1|2)$ has been described in [3]. Its generators are the even element $H$ and the pair of odd elements $v_\pm$. The graded commutation relations read

$$[H, v_\pm] = \pm\tfrac{1}{2} v_\pm \qquad [v_+, v_-] = -\frac{\sinh(\eta H)}{\sinh(2\eta)}. \tag{1.1}$$

The Casimir element $c_2(\eta)$ is

$$\begin{aligned} c_2(\eta) &= \cosh 2\eta \left(H + \frac{1}{4}\right) - \mathrm{e}^{\eta/4}\cosh\eta\left(H + \frac{1}{2}\right) ts - \frac{\mathrm{e}^{\eta/2}}{8}\sinh^{-2}(\eta/4) t^2 s^2 \\ s &= x\mathrm{e}^{-\eta H/2} v_+ \qquad t = x\mathrm{e}^{\eta H/2} v_- \qquad x = (4\sinh(\eta/2)\sinh(2\eta))^{1/2}. \end{aligned} \tag{1.2}$$

The Hopf superalgebra structure is defined by the following comultiplication, antipode and counit:

$$\begin{aligned} &\Delta v_\pm = v_\pm \otimes \mathrm{e}^{\eta H/2} + \mathrm{e}^{-\eta H/2} \otimes v_\pm \\ &\Delta H = H \otimes \mathbf{1} + \mathbf{1} \otimes H \\ &\gamma(H) = -H \qquad \gamma(v_\pm) = -\mathrm{e}^{\pm\eta/4} v_\pm \\ &\epsilon(\mathbf{1}) = 1 \qquad \epsilon(H) = \epsilon(v_\pm) = 0 \end{aligned} \tag{1.3}$$

where here and later the tensor product is also graded.

The contraction technique used in [7–9] will be applied here, giving an interesting result in the graded case as well. We present the contraction in the case of the Lie superalgebra where the even generators are $X_\pm, H$ and the odd are $v_\pm$. The algebra is given by the following graded commutators:

$$\begin{aligned} &[H, X_\pm] = \pm X_\pm \qquad [X_+, X_-] = 2H \\ &[H, v_\pm] = \pm\tfrac{1}{2} v_\pm \qquad [X_\pm, v_\mp] = v_\pm \qquad [X_\pm, v_\pm] = 0 \\ &[v_+, v_-] = -\tfrac{1}{2} H \qquad [v_\pm, v_\pm] = \pm\tfrac{1}{2} X_\pm. \end{aligned} \tag{1.4}$$

In analogy with the purely bosonic situation [9] we extend $\mathrm{osp}(1|2)$ by a central generator $Q$ and contract this extension. The scaling we shall use is the following:

$$\begin{pmatrix} b_- \\ b_+ \\ a_- \\ a_+ \\ \mathcal{N} \\ h \end{pmatrix} = \begin{pmatrix} \varepsilon & & & & & \\ & \varepsilon & & & & \\ & & \varepsilon & & & \\ & & & \varepsilon & & \\ & & & & -1 & \varepsilon^{-2} \\ & & & & 0 & 2 \end{pmatrix} \begin{pmatrix} v_+ \\ v_- \\ X_+ \\ X_- \\ H \\ Q \end{pmatrix} \tag{1.5}$$

where $\varepsilon$ is an even parameter, so that the contraction preserves the grading.

The contracted algebra reads:

$$\begin{aligned} &[\mathcal{N}, a_\pm] = \pm a_\pm \qquad [a_-, a_+] = h \\ &[\mathcal{N}, b_\pm] = \pm\tfrac{1}{2} b_\pm \qquad [a_\pm, b_\pm] = [a_\pm, b_\mp] = 0 \\ &[b_+, b_-] = -\tfrac{1}{4} h \qquad [b_\pm, b_\pm] = 0 \end{aligned} \tag{1.6}$$

$h$ being a central element. The contracted Casimir $c_2$ becomes

$$c_2 = -h\mathcal{N} + \tfrac{1}{2}(a_-a_+ + a_+a_-) + (b_-b_+ - b_+b_-). \tag{1.7}$$

We now consider the contraction of the $q$-deformed algebra. According to (1.1), maintaining the definition $X_\pm = \pm 2[v_\pm, v_\pm]$ and rescaling the quantum parameter as

$$w = 2\varepsilon^{-2}\eta \tag{1.8}$$

we see that the commutators which differ from the classical relations (1.6) are

$$[b_-, b_+] = -\frac{1}{w}\sinh\left(\frac{wh}{4}\right) \qquad [a_-, a_+] = \frac{2}{w}\sinh\left(\frac{wh}{2}\right). \tag{1.9}$$

Here we see that $(a_+, a_-, \mathcal{N})$ generate the quantum algebra $H_q(1)$ defined in [9], while $(b_+, b_-, \mathcal{N})$ give a quantum fermionic structure. The contraction of $c_2(\eta)$ is

$$c_2(w) = -\frac{2}{w}\sinh\left(\frac{wh}{2}\right)\mathcal{N} + \tfrac{1}{2}(a_-a_+ + a_+a_-) + \cosh\left(\frac{wh}{4}\right)(b_-b_+ - b_+b_-) \tag{1.10}$$

which represents the $q$-deformation of (1.7). The contractions of coproducts, antipodes and counits are then straightforward:

$$\begin{aligned}
&\Delta b_\pm = b_\pm \otimes \mathrm{e}^{wh/8} + \mathrm{e}^{-wh/8} \otimes b_\pm \\
&\Delta a_\pm = a_\pm \otimes \mathrm{e}^{wh/4} + \mathrm{e}^{-wh/4} \otimes a_\pm \\
&\Delta\mathcal{N} = \mathcal{N} \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{N} \qquad \Delta h = h \otimes \mathbf{1} + \mathbf{1} \otimes h \\
&\gamma(b_\pm) = -b_\pm \qquad \gamma(a_\pm) = -a_\pm \\
&\gamma(\mathcal{N}) = -\mathcal{N} \qquad \gamma(h) = -h \\
&\epsilon(\mathbf{1}) = 1 \qquad \epsilon(h) = \epsilon(\mathcal{N}) = \epsilon(b_\pm) = \epsilon(a_\pm) = 0.
\end{aligned} \tag{1.11}$$

We shall now turn to the representations obtained by contracting those described in [3]. The latter are summarized as follows.

$$He^l_m = \frac{m}{2}e^l_m \qquad v_\pm e^l_m = N_\pm(l,m)e^l_{m\pm 1} \tag{1.12}$$

where

$$N_+^2(l,m) = \left(\sinh(2\eta)\cosh\frac{\eta}{4}\right)^{-1} \times \begin{cases} \sinh\left(\dfrac{\eta(l+m+1)}{4}\right)\cosh\left(\dfrac{\eta(l-m)}{4}\right) & l-m \text{ odd} \\ \sinh\left(\dfrac{\eta(l-m)}{4}\right)\cosh\left(\dfrac{\eta(l+m+1)}{4}\right) & l-m \text{ even} \end{cases} \tag{1.13}$$

with $N_-(l,m) = (-1)^{l-m-1}N_+(l, m-1)$, $m = l, l-1, l-2, \ldots, -l$.

The rescaling of the generators given in (1.5) leads to the introduction of the parameters $k$ and $n$ for the contracted representations [3] defined as

$$l = \varepsilon^{-2} k \qquad l - m = n. \tag{1.14}$$

As a consequence the leading terms of $N_\pm$ are

$$N_+(k,n) = \begin{cases} \varepsilon^{-1}\sqrt{(\sinh(wk/4))/w} & n \text{ odd} \\ \frac{1}{2}\sqrt{\frac{n}{2}\cosh(wk/4)} & n \text{ even} \end{cases} \tag{1.15}$$

and

$$N_-(k,n) = \begin{cases} \frac{1}{2}\sqrt{\frac{n+1}{2}\cosh(wk/4)} & n \text{ odd} \\ -\varepsilon^{-1}\sqrt{(\sinh(wk/2))/w} & n \text{ even.} \end{cases} \tag{1.16}$$

Performing the limit $\varepsilon \to 0$ the matrix elements of $b_\pm$ and $a_\pm$ are

$$\begin{aligned} b_+ e^k_{2n+1} &= 0 \qquad b_+ e^k_{2n} = -\sqrt{\frac{\sinh(wk/4)}{w}}\, e^k_{2n+1} \\ b_- e^k_{2n+1} &= \sqrt{\frac{\sinh(wk/4)}{w}}\, e^k_{2n} \qquad b_- e^k_{2n} = 0 \end{aligned} \tag{1.17}$$

and

$$\begin{aligned} a_+ e^k_{2n-1} &= \sqrt{n\frac{\sinh(wk/2)}{w/2}}\, e^k_{2n+1} \qquad a_+ e^k_{2n-2} = \sqrt{n\frac{\sinh(wk/2)}{w/2}}\, e^k_{2n} \\ a_- e^k_{2n+1} &= \sqrt{n\frac{\sinh(wk/2)}{w/2}}\, e^k_{2n-1} \qquad a_- e^k_{2n} = \sqrt{n\frac{\sinh(wk/2)}{w/2}}\, e^k_{2n-2}. \end{aligned} \tag{1.18}$$

Moreover

$$\mathcal{N} e^k_n = \frac{n}{2} e^k_n \qquad h e^k_n = k e^k_n. \tag{1.19}$$

The $R$-matrix for $\mathrm{osp}_q(1|2)$ is given, in terms of $s$ and $t$ as in (1.2), by [3]

$$R = \exp(-2\eta H \otimes H) \sum_{i=0}^{\infty} a_i s^i \otimes t^i \tag{1.20}$$

where

$$a_i = \frac{e^{-\eta i(i+1)/8}}{(e^{-\eta} - 1)^i [i]_+!} \qquad [i]_+ = \frac{(-)^{i-1} e^{-\eta i/4} + e^{\eta i/4}}{e^{-\eta/4} + e^{\eta/4}}. \tag{1.21}$$

For $\varepsilon \to 0$ we get

$$\begin{aligned} s &\to \varepsilon w \mathrm{e}^{-wh/8} b_- \qquad t \to \varepsilon w \mathrm{e}^{wh/8} b_+ \\ s^2 &\to \frac{\varepsilon^3}{4} w^2 \mathrm{e}^{-wh/4} a_- \qquad t^2 \to -\frac{\varepsilon^3}{4} w^2 \mathrm{e}^{-wh/4} a_+ \end{aligned} \tag{1.22}$$

and

$$a_{2i} \to \frac{2^{4i}}{\varepsilon^{6i} w^{3i} i!} \qquad a_{2i+1} \to -\frac{2^{4i+1}}{\varepsilon^{6i+2} w^{3i+1} i!}. \tag{1.23}$$

The sum in the $R$-matrix is finite in the limit $\varepsilon \to 0$ and reads

$$\sum_{i=0}^{\infty} a_i s^i \otimes t^i = \exp(-w\mathrm{e}^{-wh/4} a_- \otimes \mathrm{e}^{wh/4} a_+)(1 - 2w\mathrm{e}^{-wh/8} b_- \otimes \mathrm{e}^{wh/8} b_+). \tag{1.24}$$

The exponent in front of the sum in the $R$-matrix is for $\varepsilon \to 0$ divergent. Here we can repeat what was seen for $\mathrm{H}_q(1)$ [9], namely that the divergent part is central. In fact using the scaling (1.5) we get

$$\exp(-2\eta H \otimes H) \Rightarrow \exp(-\varepsilon^{-2}(w/4) h \otimes h) \exp((w/2)(h \otimes \mathcal{N} + \mathcal{N} \otimes h)). \tag{1.25}$$

Then the limit of $R$, disregarding the central term, is

$$R = \exp((w/2)(h \otimes \mathcal{N} + \mathcal{N} \otimes h)) \exp(-w\mathrm{e}^{-wh/4} a_- \otimes \mathrm{e}^{wh/4} a_+) \times (1 - 2w\mathrm{e}^{-wh/8} b_- \otimes \mathrm{e}^{wh/8} b_+) \tag{1.26}$$

or also

$$R = \mathrm{e}^{w\Omega/2} \mathrm{e}^{\Lambda} \mathrm{e}^{\Pi} \tag{1.27}$$

where, with $\Gamma = h \otimes \mathbf{1} - \mathbf{1} \otimes h$, we have

$$\begin{aligned} \Omega &= h \otimes \mathcal{N} + \mathcal{N} \otimes h \qquad \Lambda = -w\mathrm{e}^{-w\Gamma/4}(a_- \otimes a_+) \\ \Pi &= -2w\mathrm{e}^{-w\Gamma/8}(b_- \otimes b_+). \end{aligned} \tag{1.28}$$

By direct computation it is straightforward to prove that $R$ defines a similarity transformation of $\Delta' = \sigma \circ \Delta$ into $\Delta$, where $\sigma$ is the graded permutation map, $\sigma(a \otimes b) = (-)^{p(a)p(b)} b \otimes a$. Taking into account the grading of the tensor product, the Yang–Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ is verified. We want to stress that the bosonic part of $R$ is exactly the same as that found in [9] and, because of the commutativity of bosons with fermions, the $R$-matrix without the term $\mathrm{e}^{\Lambda}$ works perfectly on the purely fermionic sector.

## 2. $\mathbf{Z}_2$-graded $q$-oscillator

Another type of contraction of $\mathrm{osp}_q(1|2)$ gives the $\mathbf{Z}_2$-graded $q$-oscillator. In this case the parameter $q$ is not rescaled: we can think of this contraction as an infinite $q$-spin limit $l \to +\infty$. Let us denote the limiting generators as

$$\begin{aligned} \zeta &= \lim_{l \to +\infty} \left( \frac{[4]_q}{[2l]_q} \right)^{1/2} v_+ \qquad \zeta^\dagger = -\lim_{l \to +\infty} \left( \frac{[4]_q}{[2l]_q} \right)^{1/2} v_- \\ N &= \lim_{l \to +\infty} 2(l - H) \end{aligned} \tag{2.1}$$

where $q = e^{-\eta/2}$ and $[x]_q = (q^x - q^{-x})(q - q^{-1})^{-1}$. The corresponding relations are:

$$[N, \zeta] = -\zeta \qquad [N, \zeta^\dagger] = \zeta^\dagger \qquad [\zeta, \zeta^\dagger] = q^{-N}. \tag{2.2}$$

In order to have relationships similar to these for the usual $q$-oscillator, we introduce the generators [10,12]

$$c = q^{N/4}\zeta \qquad c^\dagger = \zeta^\dagger q^{N/4} \tag{2.3}$$

so that

$$cc^\dagger + q^{1/2}c^\dagger c = q^{-N/2}. \tag{2.4}$$

There is a central element as for the $q$-oscillator, namely ($[\ ]_+$) as in (1.21))

$$z = (-1)^N q^{-N/2}(c^\dagger c - [N]_+). \tag{2.5}$$

It is interesting to note that there is a contraction limit of the $\mathrm{osp}_q(1|2)$ Casimir element (1.2) which does not coincide with (2.5), but is a quadratic polynomial in $z$.

One can easily construct a set of representations of the $\mathbf{Z}_2$-graded oscillator (2.3)–(2.4) parametrized by $\rho \in \mathbf{C}$ (omitting the requirements of $*$-Hermiticity connection between $c$ and $c^\dagger$). The representation is given in a $\mathbf{Z}_2$-graded linear space $l^2$, where the parity of the basis vector $f_n$ is $(-1)^n$ and where the action of $c$, $c^\dagger$, $N$ on $f_n$ reads

$$cf_n = \alpha_n f_{n-1} \qquad c^\dagger f_n = \beta_n f_{n+1} \qquad Nf_n = nf_n \tag{2.6}$$

where

$$\alpha_n\beta_n = \rho q^{n/2} - (-1)^n[n]_+ \qquad \mathrm{Spec}N = \{\ldots -2, -1, 0, 1, 2, \ldots\}. \tag{2.7}$$

Just as in the case of the $q$-oscillator, the question of the existence of a coproduct is left open.

## 3. s-$\mathbf{E}_q(2)$ superalgebra

In this section we shall describe the contraction of $\mathrm{osp}_q(1|2)$ giving rise to a graded form of the $\mathrm{E}_q(2)$ Euclidean algebra with non-symmetric coproduct, following the lines of [6, 9]. The scaling of the generators is given by

$$\mathcal{D}_\pm = \varepsilon v_\pm \qquad T_\pm = \pm\frac{\varepsilon^2}{4}X_\pm = \frac{\varepsilon^2}{2}[v_\pm, v_\pm] \tag{3.1}$$

while the generator $H$ and the deformation parameter are held fixed. As a result we obtain the relationships defining s-$\mathrm{E}_q(2)$:

$$\begin{aligned}&[\mathcal{D}_+, \mathcal{D}_-] = 0\\ &[H, \mathcal{D}_\pm] = \pm\tfrac{1}{2}\mathcal{D}_\pm \qquad \text{or} \qquad \mathrm{e}^{\eta H}\mathcal{D}_\pm = \mathcal{D}_\pm \mathrm{e}^{\eta(H\pm 1/2)}.\end{aligned} \tag{3.2}$$

The structure of the Hopf algebra is then completed by

$$\Delta(\mathcal{D}_\pm) = \mathcal{D}_\pm \otimes \mathrm{e}^{\eta H/2} + \mathrm{e}^{-\eta H/2} \otimes \mathcal{D}_\pm$$
$$\gamma(\mathcal{D}_\pm) = -\mathrm{e}^{\pm\eta/4}\mathcal{D}_\pm \qquad (3.3)$$
$$\epsilon(\mathcal{D}_\pm) = 0.$$

From (3.1) $T_\pm = \pm\mathcal{D}_\pm/u$ together with $H$ reproduce the structure of the algebra of E(2) (the coalgebra, however, is different). The representations of s-$\mathrm{E}_q(2)$ are obtained as a limit of (1.11)–(1.12) by making the appropriate scaling of $l$. Indeed, with

$$r = \varepsilon \mathrm{e}^{\eta(2l+1)/8} (\sinh(2\eta)\cosh(\eta/4))^{-1/2} \qquad (3.4)$$

the representations read

$$\mathcal{D}_+ f_n = r f_{n+1} \qquad \mathcal{D}_- f_n = (-1)^n r f_{n-1} \qquad H f_n = -\frac{n}{2} f_n . \qquad (3.5)$$

To conclude our detailed analysis of the contractions of $\mathbf{Z}_2$-graded $q$-deformed Lie algebras we want to study the analogue of the contraction to $\mathrm{E}_q(2)$ defined in [7]. To this purpose we have to introduce a fermion number operator $F$ for s-$\mathrm{E}_q(2)$. By analogy with [7], we define

$$d_1 = v_+ - (-1)^F v_- \qquad d_2 = (-1)^F v_+ + v_- \qquad (3.6)$$

with

$$[d_1, d_2] = -2\frac{\sinh(\eta H)}{\sinh(2\eta)}$$
$$[H, d_1] = \tfrac{1}{2}(-1)^F d_2 \qquad [H, d_2] = \tfrac{1}{2}(-1)^F d_1 . \qquad (3.7)$$

The rescaling

$$H \mapsto \varepsilon H \qquad d_1 \mapsto \varepsilon d_1 \qquad d_2 \mapsto d_2 \qquad \eta \mapsto \varepsilon^{-1} w \qquad (3.8)$$

defines a different structure of a $q$-deformed s-E(2). The properties of the coalgebra—e.g. the comultiplication of the fermion number operator—remain to be investigated.

We finally observe that, similarly to what has been established in the purely bosonic case [6], the results of this section support the existence of a graded analogue of $q$-special functions.

## References

[1] Kulish P P and Reshetikhin N Yu 1981 *Zap. Sem. LOMI* **101** 101
Drinfeld V G 1986 *Proc. ICM Berkeley, CA* (Providence, RI: AMS) p 798
Jimbo M 1989 *Int. J. Mod. Phys.* A **4** 3759
Reshetikhin N Yu, Takhtajan L and Faddeev L 1990 *Algebra and Analysis* **1** 193
[2] Kac V G 1977 *Adv. Math.* **26** 8
Giachetti R and Ricci R 1986 *Adv. Math.* **84** 62
[3] Kulish P P 1988 *Quantum Superalgebra* osp(2|1) RIMS-615 (Kyoto)
Kulish P P and Reshetikhin N Yu 1989 *Lett. Math. Phys.* **18** 143

Chaichian M and Kulish P P *Preprint* CERN-TH
[4] Inönu E and Wigner E P 1953 *Proc. Natl. Acad. Sci. USA* **39** 510
Celeghini E and Tarlini M 1981 *Nuovo Cimento* B **61** 265; 1981 *Nuovo Cimento* B **65** 172; 1982 *Nuovo Cimento* B **68** 133
[5] De Montigny M and Patera J 1991 *J. Phys. A: Math. Gen.* **24** 525
[6] Vaksman L L and Korogodski L I 1989 *Sov. Math. Dokl.* **39** 173
[7] Celeghini E, Giachetti R, Sorace E and Tarlini M 1990 *J. Math. Phys.* **31** 2548
[8] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 *J. Math. Phys.* **32** 1159
[9] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 *J. Math. Phys.* **32** 1155; 1990 *Contractions of quantum groups* Contribution to the First Semester 'Quantum Groups', Leningrad, in press
[10] Kulish P P 1991 *Teor. Mat. Fiz.* **86** 158
[11] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[12] Kulish P P and Damashinsky E V 1990 *J. Phys. A: Math. Gen.* **23** L415